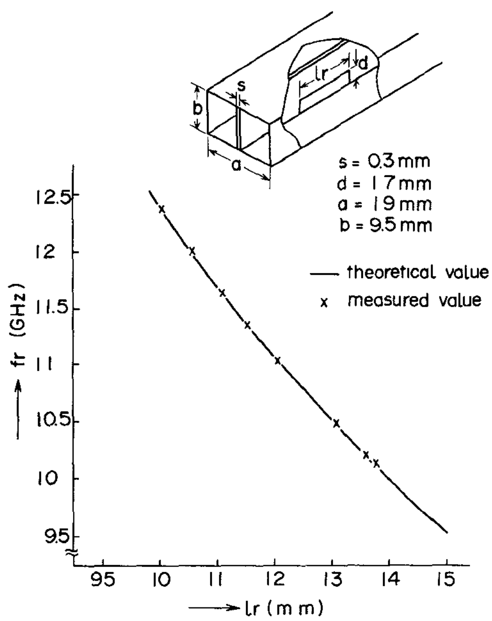


Fig. 4. Theoretical value and measured value of resonant frequency of the ridge guide resonator with planar circuit of mounted waveguide.

obtaining the resonant frequency by considering the end effect and the guided wave length of a ridge guide as (12):

$$\frac{\lambda_1^{(1)}(\omega)}{2} = l_r + 2\Delta l. \tag{12}$$

Since $\lambda_1^{(1)}$ and $\Delta l$ of (12) are the functions of $\omega$, the values of the resonant frequency were calculated by computer. The results and the measured values are shown in Fig. 4.

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# Propagation Through Hollow Cylindrical Anisotropic Dielectric Guides

B. B. CHAUDHURI AND D. K. PAUL

*Abstract*—**Hybrid mode in a circular hollow anisotropic dielectric guide is reported here. The use of such guide in fabricating gas laser is compared with its isotropic counterpart. It has been shown that proper choice of anisotropic material can increase the net gain of gas lasers.**

HOLLOW dielectric waveguides at optical and infrared frequency have found wide use in high-pressure laser oscillators and amplifiers. The waveguides are low-loss if the cross-section dimensions are many wavelengths and the interior walls have an optical finish. However, the gain of the amplifier falls approximately as the square root of the cross-sectional area, so that there is an optimum dimension for which net gain is maximum. Isotropic hollow guides have been analyzed by different authors [1], [2].

The following note analyzes hollow circular waveguide with anisotropic dielectric, where it has been shown that for the dominant hybrid mode $HE_{11}$ at 6328 Å a net gain of 1.5 dB/m over that of the isotropic guide can be achieved by proper choice of anisotropy.

Consider a hollow waveguide of radius $r_0$ with axis coinciding with the $z$-axis of $(r,\phi,z)$ coordinate system. The surrounding medium has dielectric constants $\epsilon_z$ in $z$-direction and $\epsilon_r$ in the transverse plane. Solving Maxwell's equation and applying boundary condition to the fields, we arrive at the following transcendental equation:

$$\left[\frac{1}{u}\frac{J_n'(u)}{J_n(u)} - \frac{1}{v_1}\frac{H_n'(v_1)}{H_n(v_1)}\right]\left[\frac{1}{u}\frac{J_n'(u)}{J_n(u)} - \frac{\bar{\epsilon}_z}{v_2}\frac{H_n'(v_2)}{H_n(v_2)}\right] = \left(\frac{nh}{K_0}\right)^2 \cdot \frac{\left(\frac{u^2}{a^2} - v_2^2\right)(u^2 - v_1^2)}{u^4 v_1^2 v_2^2} \tag{1}$$

where $h$ is the propagation constant in the $z$-direction and

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The authors are with the Department of Electrical Engineering, Indian Institute of Technology Kanpur, Kanpur-208 016, U.P., India.

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$a^2 = \epsilon_r/\epsilon_z$, the anisotropy parameter, with $\bar{\epsilon}_i = \epsilon_i/\epsilon_0$

$$v_1^2 = \left[\omega^2\mu_0\epsilon_r - h^2\right]r_0^2$$
$$v_2^2 = \left[\omega^2\mu_0\epsilon_z - (h/a)^2\right]r_0^2$$
$$u^2 = \left[\omega^2\mu_0\epsilon_0 - h^2\right]r_0^2 \tag{2}$$

$\epsilon_0$, $\mu_0$ are the permittivity and permeability of free space. $J_n$ and $H_n$ denote conventional Bessel and Hankel function of the $n$th order and first kind.

If $r_0/\lambda \gg 1$, $|h| \simeq K_0 = \omega\sqrt{\mu_0\epsilon_0}$. Also $v_1$, $v_2 \gg 1$ since $\bar{\epsilon}_r > 1$, $\bar{\epsilon}_z > 1$. Neglecting the second-order terms in $(u/v_i)$ where $i = 1, 2$ and using large argument approximation of Hankel function, we obtain

$$\frac{J_{n-1}(u)}{J_n(u)} \simeq \frac{ju}{2}\left[\frac{1}{v_1} + \frac{\bar{\epsilon}_z}{v_2}\right]. \tag{3}$$

Since the right-hand side of (3) is a small quantity we can assume that the solution of (3) is a perturbation $\delta_{nm}$ over $U_{nm}$ where $J_{n-1}(U_{nm}) = 0$. Hence, writing $u = U_{nm}(1+\delta_{nm})$ we get

$$\delta_{nm} \simeq -\frac{j}{2}\left(\frac{1}{v_1} + \frac{\bar{\epsilon}_z}{v_2}\right) = -\frac{j(1+\bar{\epsilon}_z a)}{2K_0 r_0(\bar{\epsilon}_r - 1)^{1/2}}.$$

Using (2) we obtain

$$h = K_0\left\{1 - \frac{1}{2}\left(\frac{U_{nm}}{r_0 K_0}\right)^2\right\} + \frac{j}{2}\left(\frac{U_{nm}}{K_0}\right)^2 \frac{1}{r_0^3}\left[\frac{1+\bar{\epsilon}_z a}{(\bar{\epsilon}_r - 1)^{1/2}}\right]. \tag{4}$$

The imaginary part of (4) accounts for the loss $L$. The loss factor $(1+\bar{\epsilon}_z a)/(\bar{\epsilon}_r - 1)^{1/2}$ is a function of anisotropy parameter and is plotted against $\bar{\epsilon}_z$ for different $a$ in Fig. 1. It is seen that the term has a minimum for $0 < a \leqslant 2$, and for $a > 1$ it is always less than that for $a = 1$. Also, the minimum shifts towards left for larger $a$.

For dominant $HE_{11}$ mode in He-Ne laser, $\lambda = 0.6328 \times 10^{-6}$ m, $U_{nm} = 2.405$. The gain is $G = A/r_0$ where the constant $A \simeq 0.00066$ dB [2]. Converting $L$ in decibels per meter the net gain is optimized with respect to $r_0$. The optimum $r_0$ is

$$r_0|_{\text{opt}} = \sqrt{3}\,(B/A)\lambda,$$

$$\text{where} \quad B = \frac{8.686}{2}\left(\frac{U_{nm}}{2\pi}\right)^2\left[\frac{1+\bar{\epsilon}_z a}{(\bar{\epsilon}_r - 1)^{1/2}}\right] \text{dB}.$$

The maximum net gain is

$$(G-L)_{\max} = \frac{2}{B^{1/2}}(A/3)^{3/2}\frac{1}{\lambda} \text{ dB/m}.$$

$(G-L)_{\max}$ is tabulated in Table I for different $a$, where $\bar{\epsilon}_z$ and $r_0|_{\text{opt}}$ are values needed for maximum net gain. It is seen that the net gain is better than in isotropic case and an improvement of $\sim$1.5 dB/m can be achieved for $a = 1.7$.

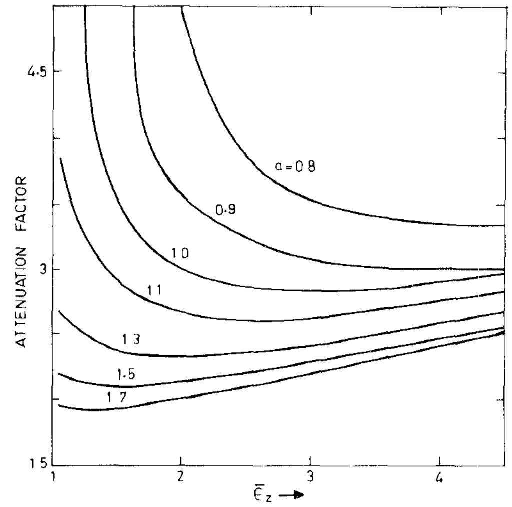


Fig. 1. Effect of anisotropy on the attenuation factor.

TABLE I
OPTIMUM DESIGN PARAMETERS IN A DIELECTRIC RESONATOR [AT $\lambda = 6328$ Å]. EFFECT OF ANISOTROPY IS INCLUDED IN $a$

| $a$ | $\bar{\epsilon}_z$ | $r_0\|_{\text{opt}}$, mm | $(G-L)_{\max}$, db/m |
|---|---|---|---|
| 1 | 3 | 0.05644 | 7.7961 |
| 1.1 | 2.6 | 0.05524 | 7.9650 |
| 1.2 | 2.2 | 0.05351 | 8.2230 |
| 1.3 | 2.0 | 0.05198 | 8.4637 |
| 1.4 | 1.7 | 0.05052 | 8.7101 |
| 1.5 | 1.6 | 0.04942 | 8.9037 |
| 1.6 | 1.4 | 0.04832 | 9.1068 |
| 1.7 | 1.3 | 0.04732 | 9.2987 |

A physical interpretation of the effect may be given as follows. The loss factor contains contribution due to TE and TM part of the hybrid wave given, respectively, by $1/(\bar{\epsilon}_r - 1)^{1/2}$ and $\bar{\epsilon}_z a/(\bar{\epsilon}_r - 1)^{1/2} = \bar{\epsilon}_r/a(\bar{\epsilon}_r - 1)^{1/2}$. For isotropic guide ($a = 1$), the TE part is the same while the TM part is $\bar{\epsilon}/(\bar{\epsilon} - 1)^{1/2}$. Absorbing $a$ in $r_0$ of imaginary part of (4) we can say that the TM wave in anisotropic guide sees a different size of the guide—larger than its physical size when $a > 1$. The loss decreases as the cube of this effective radius.

The effect can be used successfully in fabricating gas laser system. The choice of material depends on its anisotropy parameter. Larger anisotropy gives better net gain. For example, maximum net gain improvement for tourmaline ($\bar{\epsilon}_r = 2.689$, $a = 1.012$) over its isotropic counterpart ($\bar{\epsilon}_z = \bar{\epsilon}_r = \bar{\epsilon}$) is 0.031 dB/m while for calcite ($\bar{\epsilon}_r = 2.7556$, $a = 1.114$), and titania ($\bar{\epsilon}_r = 7.3411$, $a = 1.118$) the respective improvements are 0.306 dB/m and 0.346

dB/m. But calcite is preferable since the net gain (9.9857 dB/m) is better than with titania (7.4596 dB/m). This is so because it can be shown that the loss factor is minimized around $\bar{\epsilon}_r = 3$, a fact true for isotropic case also [2]. It infers that a material with $\bar{\epsilon}_r \simeq 3$ and $a \simeq 2$ is most suitable for best output.


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# Toroidal Resonator with a Conducting Separating Wall

RUDOLF DEUTSCH

***Abstract*—The exact solution of Maxwell's equations for electromagnetic waves in toroidal resonators with a separating wall was obtained. The components of the intensities of the electric and magnetic fields, the charge densities on the toroidal surface and on the separating wall, the magnetic field lines, and the dispersion relation were determined. Both the empty torus and the coaxial torus were studied. A general method to determine in an easy way the magnetic field lines from the structure of the Hertz vector is given.**

## I. INTRODUCTION

IT WAS RECENTLY shown [1]–[3] that the vectorial Helmholtz equation for electromagnetic waves in toroidal coordinates can be reduced to the scalar Helmholtz equation, and solutions of this equation for some cases important in electronics and in plasma physics were obtained. It is possible to get an exact solution with the periodicity of $4\pi$ [2]. This corresponds to an empty or coaxial torus containing a conducting separating wall (Fig. 3). In this paper, we shall study this solution in detail.

In the first part of the paper, we show that it is possible to introduce a generating function, related to the Cartesian components of the Hertz vector, and this function describes the magnetic surfaces of the stationary waves. This provides a general, easy method to construct the magnetic field lines. A series of examples of magnetic surfaces are given.

In the second part, we formulate the generating function for the exact solution with periodicity $4\pi$ and we describe the electromagnetic field in the resonators with a separating wall. The components of the electric and magnetic field's intensities, the magnetic field lines, and the charge densities on the conducting toroidal surface and on the separating wall are determined. Both the empty torus and the coaxial torus are studied.

In the third part of the paper, some particular examples are described.

## II. THE STRUCTURE OF THE MAGNETIC FIELD OF THE STATIONARY WAVES

The intensity of the magnetic field can be expressed through the Hertz vector using the well-known relation

$$\vec{B} = i\omega\epsilon_0\mu_0 \operatorname{curl} \vec{P}. \tag{1}$$

Writing the differential equations for the magnetic field lines and inserting the field components from (1), we get, for toroidal systems, the following equations:

$$\begin{aligned} dP_\rho &= \frac{\partial P_\rho}{\partial \rho}\, d\rho + \frac{\partial(\rho P_\theta)}{\partial \rho}\, d\theta + \frac{\partial}{\partial \rho}(1-\rho\cos\theta)P_\phi\, d\phi \\ d(\rho P_\theta) &= \frac{\partial P_\rho}{\partial \theta}\, d\rho + \frac{\partial(\rho P_\theta)}{\partial \theta}\, d\theta + \frac{\partial}{\partial \theta}(1-\rho\cos\theta)P_\phi\, d\phi \\ d[(1-\rho\cos\theta)P_\phi] &= \frac{\partial P_\rho}{\partial \phi}\, d\rho + \frac{\partial(\rho P_\theta)}{\partial \phi}\, d\theta + \frac{\partial}{\partial \phi}(1-\rho\cos\theta)P_\phi\, d\phi \end{aligned} \tag{2}$$

where $\rho$, $\theta$, and $\phi$ are the toroidal coordinates.

In the case of the electromagnetic waves, the components of the Hertz vector can be expressed through a

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The author is with the Institut für Theoretische Physik, Universität Innsbruck, Innsbruck, Austria.

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